

Invariant Imbedding Equations for Electromagnetic Waves in Stratified Magnetic Media: Applications to One-Dimensional Photonic Crystals

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We derive the invariant imbedding equations for plane electromagnetic waves propagating in stratified magnetic media, where both dielectric and magnetic permeabilities vary in one spatial direction in an arbitrary manner. These equations allow us to obtain the reflection and transmission coefficients of the waves and the field amplitudes inside the media exactly for any polarization and incident angle of the incoming wave by solving an initial value problem of a small number of ordinary differential equations. We apply our results to one-dimensional photonic crystals, where the periodic variations of both dielectric and magnetic permeabilities create photonic band gaps in the frequency spectrum.

The propagation of electromagnetic waves in inhomogeneous media is an important topic in various branches of physics, including optics, plasma physics, astrophysics, and condensed matter physics [1, 2, 3, 4]. Great attention has been paid to the cases where the inhomogeneity is random or periodic [1, 4, 5, 6, 7, 8, 9, 10]. The problem of electromagnetic wave propagation in random dielectric media is analogous to the Anderson localization problem of noninteracting electrons in a random potential [4, 11, 12, 13]. In recent years, much effort has been made to demonstrate the localization of light experimentally [14, 15] and to use the effect for developing the so-called random laser [16, 17]. Another important area of research has been created from an analogy with the electronic band structure problem in condensed matter physics. In media where the dielectric permeability varies periodically in space, a series of photonic band gaps can appear in their frequency spectra [12, 18, 19]. Because of the promising possibility that a number of devices based on this phenomenon can be used in various optoelectronic applications, research activity in photonic band gap structures has increased explosively in the last decade [13, 20].

A number of analytical and numerical techniques have been developed for analyzing the wave propagation in inhomogeneous media [1, 2, 3, 4, 5, 6, 7, 21, 22]. In the most general cases where the inhomogeneity is three-dimensional, the exact solution of the problem is extremely hard to obtain, and one is forced to use some approximation scheme. In the cases where the inhomogeneity is one-dimensional, however, there exist several theoretical methods that allow (at least numerically) exact solutions of the problem in some simple situations. The invariant imbedding method is a powerful and most versatile tool for handling one-dimensional situations [7, 23, 24, 25, 26, 27]. It can be used to obtain exact solutions for the reflection and transmission coefficients of incoming waves and the electric and magnetic

field amplitudes inside arbitrarily inhomogeneous media. When the inhomogeneity is random, it can be used to obtain the exact disorder-averaged reflection and transmission coefficients and field amplitudes [27]. The invariant imbedding method can also be used to study the wave propagation in nonlinear dielectric media [23, 26] and the propagation of several coupled waves [24] in an exact manner.

In the present letter, we apply the invariant imbedding method to cases where both the dielectric and magnetic permeabilities vary in one direction in space in an arbitrary manner and derive a set of first-order ordinary differential equations called the invariant imbedding equations. To the best of our knowledge, these equations have never been derived before. We solve the initial value problem of the invariant imbedding equations numerically to obtain the exact reflection and transmission coefficients and field amplitudes. We apply our results to the cases where one-dimensional periodic variations of both the dielectric and magnetic permeabilities create photonic band gaps in the frequency spectrum. This situation can be realized in one-dimensional magnetic photonic crystals made of alternating layers of dielectric and ferrite films [28, 29, 30, 31].

We are interested in the propagation of a plane, monochromatic, and linearly-polarized electromagnetic wave of angular frequency ω and vacuum wave number $k_0 = \omega/c$, where c is the speed of light in vacuum. The wave is assumed to be incident from a homogeneous region on a layered or stratified medium, where the dielectric permeability ϵ and the magnetic permeability μ (and therefore the refractive index $n = \sqrt{\epsilon\mu}$ and the wave impedance $Z = \sqrt{\mu/\epsilon}$) vary only in one direction in space. We take this direction as the z -axis and assume the inhomogeneous, but isotropic, medium lies in $0 \leq z \leq L$. Without loss of generality, we assume that the wave propagates in the xz -plane. Since the medium is uniform in the x -direction, the dependence on x can be

taken as being through a factor e^{iqx} , with q being a constant. When $q = 0$, the wave passes through the medium normally. If $q \neq 0$, the wave is said to pass obliquely.

For $q \neq 0$, we have to distinguish two independent cases of polarization. In the first case, the electric field vector is perpendicular to the xz -plane. Then one can show easily from Maxwell's equations that the complex amplitude of the electric field, $E = E(z)$, satisfies

$$\frac{d^2 E}{dz^2} - \frac{1}{\mu(z)} \frac{d\mu}{dz} \frac{dE}{dz} + [k_0^2 \epsilon(z) \mu(z) - q^2] E = 0. \quad (1)$$

This type of wave is known as an s (or TE) wave. In the other case, the magnetic field vector is perpendicular to the xz -plane. Then the complex amplitude of the magnetic field, $H = H(z)$, satisfies

$$\frac{d^2 H}{dz^2} - \frac{1}{\epsilon(z)} \frac{d\epsilon}{dz} \frac{dH}{dz} + [k_0^2 \epsilon(z) \mu(z) - q^2] H = 0. \quad (2)$$

This type of wave is called p (or TM) wave. We note that there is a simple symmetry between Eqs. (1) and (2): Eq. (2) is obtained from Eq. (1) by exchanging ϵ and μ and replacing E by H . From now on, we will consider only the s wave. The results for the p wave follow from those for the s wave in a trivial manner.

We assume that the wave is incident from the region where $z > L$ and is transmitted to the region where $z < 0$. The dielectric and magnetic permeabilities are assumed to be given by

$$\epsilon(z) = \begin{cases} \epsilon_1 & \text{if } z > L \\ \epsilon_R(z) + i\epsilon_I(z) & \text{if } 0 \leq z \leq L \\ \epsilon_2 & \text{if } z < 0 \end{cases}, \quad \mu(z) = \begin{cases} \mu_1 & \text{if } z > L \\ \mu_R(z) + i\mu_I(z) & \text{if } 0 \leq z \leq L \\ \mu_2 & \text{if } z < 0 \end{cases}, \quad (3)$$

where ϵ_1 , ϵ_2 , μ_1 , and μ_2 are real constants and $\epsilon_R(z)$, $\epsilon_I(z)$, $\mu_R(z)$, and $\mu_I(z)$ are *arbitrary* real functions of z . When θ is defined as the angle of incidence, the constant q in Eqs. (1) and (2) is equal to $\sqrt{\epsilon_1 \mu_1} k_0 \sin \theta$.

We consider a plane wave of unit magnitude $\tilde{E}(x, z) = E(z)e^{iqx} = e^{ip(L-z)+iqx}$, where $p = \sqrt{\epsilon_1 \mu_1} k_0 \cos \theta$, incident on the medium from the right. The quantities of main interest are the complex reflection and transmission coefficients, $r = r(L)$ and $t = t(L)$, defined by the wave functions outside the medium:

$$\tilde{E}(x, z) = \begin{cases} e^{ip(L-z)+iqx} + r(L)e^{ip(z-L)+iqx}, & z > L \\ t(L)e^{-ip'z+iqx}, & z < 0 \end{cases}, \quad (4)$$

where $p' = \sqrt{\epsilon_2 \mu_2} k_0 \cos \theta'$ and θ' is the angle that outgoing waves make with the negative z -axis.

We generalize the invariant imbedding method developed in Ref. 23 to the cases where both ϵ and μ are inhomogeneous. Let us consider the E field in Eq. (1) as

a function of both z and L : $E = E(z; L)$. Starting from the crucial observation that the boundary value problem of the wave equation, Eq. (1), can be transformed into an integral equation

$$E(z; L) = g(z; L) + \frac{ip}{2} \int_0^L dz' g(z; z') \times \left\{ \frac{\epsilon(z')}{\epsilon_1} - \frac{\mu(z')}{\mu_1} + \frac{q^2}{p^2} \left[\frac{\epsilon(z')}{\epsilon_1} - \frac{\mu_1}{\mu(z')} \right] \right\} E(z'; L),$$

$$g(z; z') = \exp \left[ip \operatorname{sgn}(z - z') \int_{z'}^z dz'' \frac{\mu(z'')}{\mu_1} \right], \quad (5)$$

we derive

$$\frac{\partial E(z; L)}{\partial L} = a(L) E(z; L), \quad (6)$$

where

$$a(L) = ip \frac{\mu(L)}{\mu_1} + \frac{ip}{2} \left\{ \frac{\epsilon(L)}{\epsilon_1} - \frac{\mu(L)}{\mu_1} + \frac{q^2}{p^2} \left[\frac{\epsilon(L)}{\epsilon_1} - \frac{\mu_1}{\mu(L)} \right] \right\} E(L; L). \quad (7)$$

Using this equation, we obtain *exact* differential equations satisfied by r and t :

$$\begin{aligned} \frac{dr(L)}{dL} &= 2i\sqrt{\epsilon_1 \mu_1} k_0 \cos \theta \frac{\mu_R(L) + i\mu_I(L)}{\mu_1} r(L) \\ &+ \frac{i}{2} \sqrt{\epsilon_1 \mu_1} k_0 \cos \theta \left[\frac{\epsilon_R(L) + i\epsilon_I(L)}{\epsilon_1} - \frac{\mu_R(L) + i\mu_I(L)}{\mu_1} \right. \\ &\quad \left. + \frac{\epsilon_R(L) + i\epsilon_I(L)}{\epsilon_1} \tan^2 \theta - \frac{\mu_1}{\mu_R(L) + i\mu_I(L)} \tan^2 \theta \right] \\ &\quad \times [1 + r(L)]^2, \\ \frac{dt(L)}{dL} &= i\sqrt{\epsilon_1 \mu_1} k_0 \cos \theta \frac{\mu_R(L) + i\mu_I(L)}{\mu_1} t(L) \\ &+ \frac{i}{2} \sqrt{\epsilon_1 \mu_1} k_0 \cos \theta \left[\frac{\epsilon_R(L) + i\epsilon_I(L)}{\epsilon_1} - \frac{\mu_R(L) + i\mu_I(L)}{\mu_1} \right. \\ &\quad \left. + \frac{\epsilon_R(L) + i\epsilon_I(L)}{\epsilon_1} \tan^2 \theta - \frac{\mu_1}{\mu_R(L) + i\mu_I(L)} \tan^2 \theta \right] \\ &\quad \times [1 + r(L)] t(L). \end{aligned} \quad (8)$$

These invariant imbedding equations are supplemented with the initial conditions for r and t , which are obtained using the well-known Fresnel formulas:

$$r(0) = \frac{\mu_2 \sqrt{\epsilon_1 \mu_1} \cos \theta - \mu_1 \sqrt{\epsilon_2 \mu_2 - \epsilon_1 \mu_1 \sin^2 \theta}}{\mu_2 \sqrt{\epsilon_1 \mu_1} \cos \theta + \mu_1 \sqrt{\epsilon_2 \mu_2 - \epsilon_1 \mu_1 \sin^2 \theta}},$$

$$t(0) = \frac{2\mu_2 \sqrt{\epsilon_1 \mu_1} \cos \theta}{\mu_2 \sqrt{\epsilon_1 \mu_1} \cos \theta + \mu_1 \sqrt{\epsilon_2 \mu_2 - \epsilon_1 \mu_1 \sin^2 \theta}}. \quad (9)$$

For given values of ϵ_1 , ϵ_2 , μ_1 , μ_2 , k_0 (or ω), and θ and for arbitrary functions $\epsilon_R(L)$, $\epsilon_I(L)$, $\mu_R(L)$, and $\mu_I(L)$, we solve the nonlinear ordinary differential equations in Eq. (8) numerically, using the initial conditions in Eq. (9),

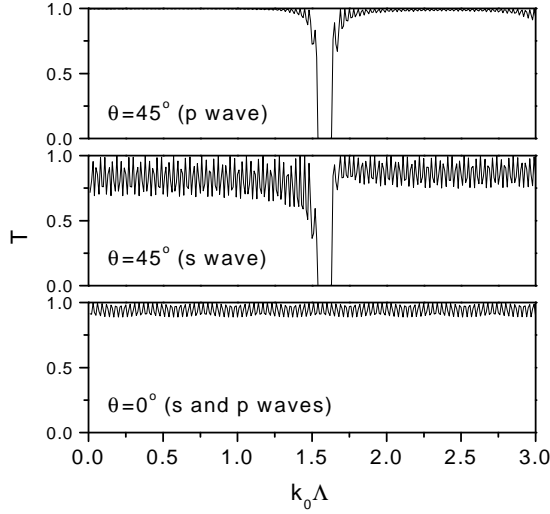


FIG. 1: Transmittance spectrum of a 100-period array of alternating layers with $\epsilon_a = 4$, $\mu_a = 2$, $\epsilon_b = 2$, $\mu_b = 1$, and $d_a = d_b = \Lambda/2$.

and obtain the reflection and transmission coefficients r and t as functions of L . The reflectivity \mathcal{R} and the transmissivity \mathcal{T} are given by

$$\mathcal{R} = |r|^2, \quad \mathcal{T} = \frac{\mu_1 \sqrt{\epsilon_2 \mu_2 - \epsilon_1 \mu_1 \sin^2 \theta}}{\mu_2 \sqrt{\epsilon_1 \mu_1 \cos \theta}} |t|^2. \quad (10)$$

If the permeabilities have no imaginary parts, the quantities \mathcal{R} and \mathcal{T} satisfy the law of conservation of energy, $\mathcal{R} + \mathcal{T} = 1$.

The invariant imbedding method can also be used in calculating the field amplitude $E(z)$ inside the inhomogeneous medium. We rewrite Eq. (6) as

$$\begin{aligned} \frac{\partial E(z; l)}{\partial l} = & i \sqrt{\epsilon_1 \mu_1} k_0 \cos \theta \left\{ \frac{\mu_R(l) + i \mu_I(l)}{\mu_1} \right. \\ & + \frac{1}{2} \left[\frac{\epsilon_R(l) + i \epsilon_I(l)}{\epsilon_1} - \frac{\mu_R(l) + i \mu_I(l)}{\mu_1} \right. \\ & + \left. \frac{\epsilon_R(l) + i \epsilon_I(l)}{\epsilon_1} \tan^2 \theta - \frac{\mu_1}{\mu_R(l) + i \mu_I(l)} \tan^2 \theta \right] \\ & \left. \times [1 + r(l)] \right\} E(z; l). \end{aligned} \quad (11)$$

For a given z ($0 < z < L$), the field amplitude $E(z; L)$ is obtained by integrating this equation from $l = z$ to $l = L$ using the initial condition $E(z; z) = 1 + r(z)$.

We demonstrate the validity and utility of our invariant imbedding equations by applying them to simple one-dimensional photonic crystals, where both ϵ and μ vary periodically in one direction in space. More specifically, we consider periodic arrays of alternating layers, characterized by the dielectric permeabilities ϵ_a and ϵ_b , the magnetic permeabilities μ_a and μ_b , and the thicknesses d_a and d_b . We assume that the imaginary parts of the

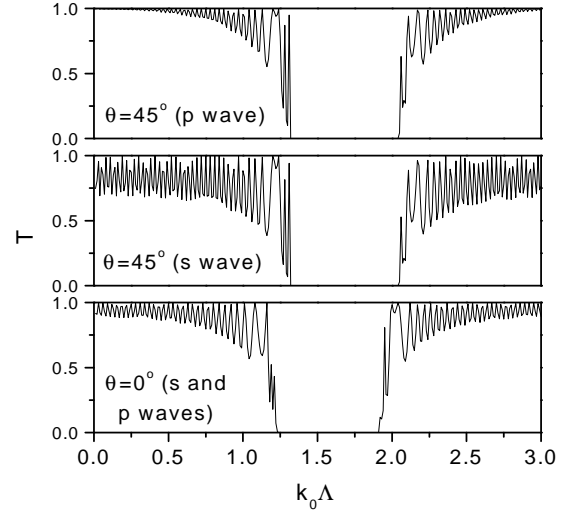


FIG. 2: Transmittance spectrum of a 100-period array of alternating layers with $\epsilon_a = 4$, $\mu_a = 1$, $\epsilon_b = 2$, $\mu_b = 2$, and $d_a = d_b = \Lambda/2$.

permeabilities are zero and the ambient medium is vacuum. In the limit where the length of the system, L , diverges to infinity, one can find the exact positions of the photonic band gaps by computing the roots of the analytical expression [31]

$$\cos \beta_a \cos \beta_b - \frac{1}{2} \left(\frac{h_a}{h_b} + \frac{h_b}{h_a} \right) \sin \beta_a \sin \beta_b + 1 = 0, \quad (12)$$

where $n_i = \sqrt{\epsilon_i \mu_i}$, $Z_i = \sqrt{\mu_i / \epsilon_i}$ ($i = a, b$) and

$$\begin{aligned} h_i = & \begin{cases} \sqrt{n_i^2 - \sin^2 \theta} / (n_i Z_i) & \text{for the } s \text{ wave} \\ Z_i \sqrt{n_i^2 - \sin^2 \theta} / n_i & \text{for the } p \text{ wave} \end{cases}, \\ \beta_i = & k_0 d_i \sqrt{n_i^2 - \sin^2 \theta}. \end{aligned} \quad (13)$$

In Fig. 1, we plot the transmittance T ($\equiv |t|^2$) of s and p waves impinging on a 100-period array with $\epsilon_a = 4$, $\mu_a = 2$, $\epsilon_b = 2$, $\mu_b = 1$, and $d_a = d_b$ as a function of $k_0 \Lambda$ where $\Lambda = d_a + d_b$. In this case, the refractive index alternates periodically between $n_a = 2\sqrt{2}$ and $n_b = \sqrt{2}$, whereas the wave impedance is uniform. We find that there is no photonic band gap when the wave is normally incident. For obliquely incident waves, small gaps are created. Furthermore, we observe that the band gaps for the s wave agree precisely with those for the p wave in their sizes and positions. When $\theta = 45^\circ$ and $L \rightarrow \infty$, we find from Eq. (12) that the lowest band gap exists for $1.54 < k_0 \Lambda < 1.63$, which is fully consistent with our numerical result obtained using the invariant imbedding method.

In Fig. 2, we plot T for the case of a 100-period array with $\epsilon_a = 4$, $\mu_a = 1$, $\epsilon_b = 2$, $\mu_b = 2$, and $d_a = d_b$. In this case, the wave impedance alternates periodically between $Z_a = 1/2$ and $Z_b = 1$, while the refractive index

is uniform. We find that large band gaps are formed for all incident angles. Similarly to the previous case, the band gaps for the s wave agree with those for the p wave in their sizes and positions. When $\theta = 0^\circ$ and 45° , the first gaps are predicted to exist for $1.23 < k_0\Lambda < 1.91$ and for $1.32 < k_0\Lambda < 2.04$, respectively, in agreement with our result.

From the two examples discussed above, we conclude that the wave impedance, not the refractive index, is a crucial parameter mainly responsible for the creation of photonic band gaps, as has been extensively discussed in Refs. 29 to 31. The equality of the s and p wave gaps is a special feature that works only when either the refractive index or the wave impedance is uniform. It is easy to verify this by examining Eq. (12).

It is straightforward to apply our equations to more realistic cases where the imaginary parts of the permeabilities are nonzero. They can also be applied to cases where the parameters contain randomly-varying components. This versatility of the invariant imbedding method makes it a very useful and convenient tool in designing practical optical devices for various application purposes.

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